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A Tractable Mean Field Game Model for the Analysis of Crowd Evacuation Dynamics

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***Abstract*— A fixed time horizon evacuation model for a large population of confined agents is proposed within the mean field games framework. Unlike previously proposed models relying on simulations or partial differential equation analysis, the proposed model remains linear in the individual agent dynamics and quadratic in the cost functions, ultimately dictating the agents' motion. We use negative definite matrices in the cost function to reflect congestion effect during evacuations. This leads to Riccati equations that generically display a finite escape time. Therefore, we develop an existence theory for the infinite population mean field game based equilibrium dynamics, and establish its ϵ-Nash property for a large but finite population of agents. Simulation results illustrating the numerical behavior of the model are presented with stress effect and different social interaction scenarios such as congestion and crowd following behavior.**

I. INTRODUCTION

Conceiving pedestrian evacuation strategies is a critical task during the design process of facilities such as subways and tunnels or the organization of outdoor events. Despite severe precautionary measures, numerous stampedes are reported yearly [1]. This appears to be mainly due to the ineffectiveness of evacuation plans. Thus, a deeper understanding of pedestrian crowd behavior is essential. Recently, multidisciplinary research by psychologists [2], computer scientists [3], and mathematicians [4] has aimed at a better understanding and prediction of pedestrian crowd behavior.

We can distinguish two main classes of mathematical models of population dynamics: microscopic [5] and macroscopic [6]. In the former group, the population is perceived as a set of interacting individuals, whereas in the latter, it is perceived as a continuum. The advantage of the microscopic approach over its macroscopic counterpart is its modeling flexibility. The population can be either homogeneous or heterogeneous, and the crowd model can be designed so that it replicates the experimentally observed pedestrian behaviors. However, this modeling flexibility comes at the price of limited computational scalability, whereas the complexity of macroscopic models is independent of the number of individuals. Several macroscopic approaches have been proposed in the literature, such as kinetic models inspired by fluid dynamics [7], the Hughes model [8], and Mean Field Game (MFG) models [9], [10]. The MFG theory assumes a large number of rational agents with weak coupling, which interact through the population's distribution. Applications of the MFG theory include energy grids, opinion formation, finance, and crowd management [11]. Our interest here is in further extending the MFG-based analysis of crowd dynamics. Several authors have modeled crowd behavior using the MFG paradigm [12]. Lachapelle et al. [13] proposed a two-population MFG-type based model that integrates the congestion and crowd following behavior. Aurell et al. [14] extended this model to include non-local agent interactions. In a later work, Aurell et al. [15] distinguished between two groups of agents: 'tagged pedestrians' who represent, for instance, firefighters and medics, and 'ordinary pedestrians' who are fleeing given emergencies. Djehiche et al. [16] addressed the evacuation process in a multi-level building under the MFG approach. One of the shortcomings of these models is that, because of assumed non linear interaction effects, they require sophisticated PDE based calculations to produce the individuals' laws of motion during an evacuation scenario. In that context, we propose a formulation relying on a linear quadratic framework, which leads to simple yet realistic sets of control laws.

Our model is an extension of the collective choice model (CCM) introduced by Salhab et al. [17] to a situation of multi-exit crowd evacuation. The CCM model was initially used to characterize time-constrained collectively influenced choices, e.g., in elections, when individuals feel the pressure of the mean opinion around them. However, while the mean state trajectory attracts the agents in the CCM model, it repels them in our proposed crowd evacuation model, since agents tend to avoid congested areas in that case. Thus, unlike the CCM model, which used positive penalty weights in a linear quadratic framework, our extension requires the introduction of *negative definite weight matrices* in the running cost function, while the input penalty matrix remains positive definite. Unfortunately, a negative running cost may lead to an unrealistic scenario where agents could achieve infinitely negative costs by escaping to infinity. This corresponds to a finite escape time in certain Riccati equations [18]. Therefore, beyond proposing a novel and more tractable

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MFG model for crowd evacuation, our contribution is in identifying sufficient conditions for the well-posedness of the proposed model. We also provide an existence proof of an asymptotic MFG equilibrium for our crowd model, which had to be modified relative to that associated with the CCM model [17]. Finally, We study the influence of stress, which is a prominent psychological factor affecting crowd behavior during an evacuation.

The rest of the paper is organized as follows. In Section II, we define the finite population model and heuristically hypothesize its corresponding infinite population counterpart. In Section III, given an arbitrary continuous mean population trajectory, we develop the Riccati equation corresponding to a generic agent's best response. Section IV then provides sufficient conditions on the time horizon length T and weight parameters in the cost for the existence of an optimal control. In Section V, we characterize the destinations' domains of attraction for an arbitrary assumed mean state continuous trajectory and probability distribution for the initial conditions. In Section VI, following [17], we prove the existence of an equilibrium for the posited infinite population game and its ϵ-Nash property in the finite case. In Section VII, we design several simulations to illustrate the crowd's behavior with given stress levels and social interaction scenarios. Section VIII concludes the paper.

II. MATHEMATICAL MODEL

A. Scalar finite population model

To motivate the use of a negative running cost in our proposed model, we first start by presenting a scalar finite horizon ($[0, T]$) non-cooperative game involving a finite number N of agents with simple integrator dynamics and randomly spread over a region of the space. We assume that there exist l exits and that agents try to avoid congestion by maximizing their mean square interdistance (i.e. $\Delta x_{i,p} = x_i - x_p$ for arbitrary agents i and p). The proposed cost function and dynamic (kinetic) model for the i^{th} generic agent can be written as follows:

$$\begin{aligned} \min_{j\in\{1..l\}} \inf_{u_i(.)} \int_0^T & -\frac{1}{N}\sum_{p=1}^{N}(x_i - x_p)^2 + R_u u_i^2 dt \\ & + M(x_i - d_j)^2 \\ \text{s.t.} \qquad & \frac{d}{dt}x_i = u_i, \;\; x_i(0) = x_{i,0}. \end{aligned}$$

where x_i and x_p are the i^{th} and the p^{th} agent's position respectively, $u_i(.)$ is the i^{th} agent control policy assumed in principle to be a feedback on its state and that of all other agents, d_j designates the j^{th} exit position, R_u and M are positive control and terminal cost penalty weights respectively. Let $\bar{X}^{(N)}(t) = \frac{1}{N}\sum_{p=1}^{N} x_p(t)$ and ${\sigma_X^2}^{(N)}(t) = \frac{1}{N}\sum_{p=1}^{N}(x_p(t) - \bar{X}^{(N)}(t))^2$ be the population's empirical mean and variance respectively. Then, by adding and subtracting $\bar{X}^{(N)}$, the generic agent's running cost function L_i can be rewritten as follows:

$$\begin{aligned} & L_i\left(x_i, u_i, \bar{X}^{(N)}\right) \\ & = -\left(x_i - \bar{X}^{(N)}\right)^2 - \frac{1}{N}\sum_{p=1}^{N}\left(x_p - \bar{X}^{(N)}\right)^2 + R_u u_i^2 \end{aligned}$$

As $N \to \infty$, assuming that agents' initial states are independent and identically distributed random variables and as is usual in MFG based analysis that the coupling of agent states through their cost functions vanishes, the running cost becomes $L_i(x_i, u_i, \bar{X}) = -(x_i - \bar{X})^2 - \sigma_X^2 + R_u u_i^2$, where $t \mapsto \bar{X}(t)$ and $t \mapsto \sigma_X^2(t)$ are respectively the mean and variance of the state of a generic agent, now considered as given continuous deterministic trajectories. Thus, for an infinite population the evacuation model for a generic agent becomes

$$\begin{aligned} \min_{j\in\{1..l\}} \inf_{u_i(.)} \int_0^T & -(x_i - \bar{X})^2 - \sigma_X^2 + R_u u_i^2 dt \\ & + M(x_i - d_j)^2 \\ \text{s.t.} \qquad & \frac{d}{dt}x_i = u_i, \;\; x_i(0) = x_{i,0}. \end{aligned} \tag{1}$$

In the case of a non-cooperative MFG, the population variance can be considered as a frozen exogenous input that does not depend on the agent's control nor affects its decisions. Thus, when computing the agent's best response, the term σ_X^2 can be dropped. We also note that the state-related terms in the running cost are negative. This reflects the fact that agents tend to maximize their inter-distances to achieve freer motion. Based on these observations, we propose in the next subsection a more general multidimensional model as it can capture agents psychological attitudes (e.g., stressed, or not stressed) as well as their social interactions (i.e., congestion avoidance versus crowd following behavior).

B. Multidimensional infinite population evacuation model

We assume that the agents' initial states are independent and identically distributed. $\mathcal{P}_0$ denotes agents' initial states distribution. $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$ are respectively the generic agent's position and control vectors. $\bar{X}(t)$ is the population mean at time t. Then, the agent's best response to a given population mean trajectory $\bar{X}(t)$ can be obtained by solving the following optimal control problem

$$\begin{aligned} & \min_{j\in\{1..l\}} \inf_{u(.)} \int_0^T \{-(x - \bar{X})^T R_x (x - \bar{X}) + (x - d_j)^T R_d \\ & (x - d_j) + u^T R_u u\} dt + (x(T) - d_j)^T M (x(T) - d_j) \\ & \text{s.t.} \qquad \frac{d}{dt}x = Ax + Bu, \;\; x(0) = x_0. \end{aligned} \tag{2}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, the pair (A,B) is assumed controllable. $R_x, R_d \in \mathbb{R}^{n\times n}$ and $M, R_u \in \mathbb{R}^{m\times m}$ are symmetric cost matrices. R_u and M are positive definite.

For the simulations in Section VII, we choose R_x to be either a positive semi-definite or a negative semi-definite matrix which reflects, respectively, congestion avoidance or crowd following behavior. Furthermore, a positive semi-definite matrix R_d is associated with what we designate as stress effect. It is meant to simulate the psychological pressure that an agent feels as long as it remains far from its candidate destination. We denote by $J_{j,M}(x_0,u,\bar{X})$ the cost for the agent starting at x_0 and applying the control $u(t), t \in [0,T]$. Finally, we denote by $\mathrm{Ev}_{d_j,M}(x_0,\bar{X})$ the evacuation problem associated with the destination d_j, the terminal penalty matrix M, the population mean $\bar{X}$, and the agent's initial position x_0.

III. Identifying the Agent's Best Response

The determination of a generic agent's best response turns into an optimal control calculation whenever one considers the infinite population problem. In fact, it reduces to a local state feedback control law whenever the mean trajectory $\bar{X}$ is considered given. In this section, we fix the destination and consider the resulting optimization problem for each destination $\mathrm{Ev}_{d_j,M}(x_0,\bar{X})$. We assume that the control horizon length is such that the optimal control for $\mathrm{Ev}_{d_j,M}(x_0,\bar{X})$ exists for all $x_0 \in \mathbb{R}^n$. Then, each agent chooses the destination with the least cost. We define $V_{j,M}$ and $\mathcal{H}$ the value function and the Hamiltonian associated with the cost $J_{j,M}$:

$$\mathcal{H}(x,u,\bar{X},\nabla_x V_{j,M},t) = -(x-\overline{X})^T R_x (x-\overline{X}) \quad (3)$$
$$+ (x-d_j)^T R_d (x-d_j) + u^T R_u u + \nabla_x V_{j,M}^T (Ax+Bu)$$

Using the dynamic programming approach, we derive the associated Hamilton-Jacobi-Bellman equation, which can be written as follows:

$$\frac{\partial}{\partial t} V_{j,M}(x,\bar{X},t) + \min_u \{\mathcal{H}(x,u,\bar{X},\frac{\partial}{\partial x}V_{j,M},t)\} = 0$$
$$\text{where } V_{j,M}(x,\bar{X},T) = (x-d_j)^T M (x-d_j) \quad (4)$$

We search for a quadratic value function

$$V_{j,M}(x,\bar{X},t) = x^T\varphi(t)x + \psi_{d_j}^T(t)x + \chi_{d_j}(t) \quad (5)$$

where $\forall t \in [0,T],\ \varphi(t) \in \mathbb{R}^{n\times n}$ is symmetric, $\psi_{d_j}(t) \in \mathbb{R}^n$ and χ_{d_j} is a scalar. Since R_u is positive definite, $\mathcal{H}$ is strictly convex, and thus, the optimal control, if it exists, is unique. By setting equation $\frac{\partial}{\partial u}\mathcal{H} = 0$, the optimal control $u^*_{d_j,M}$ should satisfy

$$u^*_{d_j,M} = -R_u^{-1}B^T\varphi x - \frac{R_u^{-1}B^T}{2}\psi_{d_j} \quad (6)$$

which gives for the HJB equation (4)

$$\begin{aligned}&x\frac{d\varphi}{dt}x + \frac{d\psi_{d_j}^T}{dt}x + \frac{d\chi_{d_j}}{dt} = \\ &x^T(R_x - R_d - \varphi A - A^T\varphi + \varphi B R_u^{-1}B^T\varphi)x \\ &+ (2d_j^T R_d - 2\bar{X}^T R_x - \psi_{d_j}^T A + \psi_{d_j}^T B R_u^{-1} B^T \varphi)x \\ &+ \bar{X}^T R_x \bar{X} - d_j^T R_d d_j + \frac{1}{4}\psi_{d_j}^T B R_u^{-1} B^T \psi_{d_j}\end{aligned}$$

Thus, φ, ψ_{d_j} and χ_{d_j} should satisfy

$$\frac{d}{dt}\varphi = \varphi S\varphi - \varphi A - A^T\varphi - Q \quad (7)$$
$$\frac{d}{dt}\psi_{d_j} = (\varphi^T S - A^T)\psi_{d_j} + 2(R_d d_j - R_x\bar{X}) \quad (8)$$
$$\frac{d}{dt}\chi_{d_j} = \frac{1}{4}\psi_{d_j}^T S\psi_{d_j} + \bar{X}^T R_x\bar{X} - d_j^T R_d d_j \quad (9)$$

where $\varphi(T) = M, \psi_{d_j}(T) = -2Md_j, \chi_{d_j}(T) = d_j^T M d_j$ and $S = BR_u^{-1}B^T, Q = R_d - R_x$. Thus, the weights ψ_{d_j} and χ_{d_j} exist if a solution to the Riccati equation (7) exists over the interval [0,T]. Using a completion of squares argument, we prove that $u^*_{d_j,M}$ is an optimal control for $\mathrm{Ev}_{d_j,M}(x_0,\bar{X})$.

IV. Conditions for the Existence of a Minimizing Control

In order to gain insight into the properties of the Riccati equation (7) and the existence of optimal control for a generic agent, we start by analyzing the scalar case, which is analytically tractable. Subsequently, we extend the results for a decoupled form of the multidimensional case. Then, we provide a theorem for a more general case.

A. Scalar Case

The scalar evacuation model is as follows:

$$\begin{aligned}\min_{j\in\{1..l\}} \inf_{u(.)} \int_0^T &\{-R_x(x-\bar{X})^2 + R_d(x-d_j)^2 + R_u u^2\}dt \\ &+ M(x(T)-d_j)^2 \\ \text{s.t. } &\frac{d}{dt}x = u,\ \ x(0) = x_0. \end{aligned} \quad (10)$$

Thus, the scalar Riccati equation for the coefficient φ is:

$$\frac{d}{dt}\varphi = \frac{1}{R_u}\varphi^2 + (R_x - R_d),\ \ \varphi(T) = M, \quad (11)$$

where $\varphi, R_x, R_d \in \mathbb{R}$ and $M, R_u \in \mathbb{R}_+$. This equation can be solved explicitly to obtain

$$\varphi(t) = \begin{cases} v\tan(p_1 t + q_1), & \text{if } R_d < R_x \\ \frac{1}{p_2 t + q_2}, & \text{if } R_d = R_x \\ v\tanh(-p_1 t + q_3), & \text{if } R_d > R_x, M < v \\ v\coth(-p_1 t + q_4), & \text{if } R_d > R_x, M > v, \end{cases} \quad (12)$$

where

$$p_1 = \sqrt{\frac{|R_x - R_d|}{R_u}}, p_2 = -\frac{1}{R_u}, v = \sqrt{R_u|R_x - R_d|}$$
$$q_1 = \tanh^{-1}(\frac{M}{v}) - Tp_1, q_2 = \frac{1}{R_u}T + \frac{1}{M}$$
$$q_3 = -p_1 T + \tanh^{-1}(\frac{M}{v}), q_4 = -p_1 T + \coth^{-1}(\frac{M}{v})$$

We conclude that the only case where the Riccati equation does not admit a solution is when $R_d < R_x$ and $T_{esc} = (\frac{\pi}{2} + \arctan(\frac{M}{v}))/p_1 \leq T$. Thus, in case $R_d < R_x$, the game time horizon T should be less than T_{esc}.

B. Multidimensional Case

Now, we extend the scalar results to the multidimensional problem. As in the scalar case, when $Q \succeq 0$ (i.e. $R_x \preceq R_d$), there exists a solution for the Riccati differential equation (7) for all time horizon $T \geq 0$ [19, Theorem 4.1.6]. We now consider the case where $A = 0$ and matrices S, Q and M are diagonal. In this case, the system of Riccati differential equations can be decoupled into n independent scalar Riccati equations. Thus, the result obtained in the scalar case applies and we have the following theorem:

***Theorem** 4.1:* If $A = 0$, the matrices S, Q, and M are diagonal, and $\exists\{1..r\} \in \{1..n\}$ such that $Q_{i,i} < 0$ $(i.e.\ [R_x]_{i,i} < [R_d]_{i,i}), \forall i \in \{1..r\}$, then, the Riccati differential equation (7) admits a continuous solution if and only if the time horizon $T < T_{esc} = \min_{i\in\{1,..,r\}}\{\frac{1}{\sqrt{|Q_{i,i}|S_{i,i}}}(\frac{\pi}{2} + \arctan(M_{i,i}\sqrt{\frac{S_{i,i}}{|Q_{i,i}|}}))\}$.

For a more general case, we have the following theorem:

***Theorem** 4.2:* The following statements hold:

1) The Riccati equation (7) admits an equilibrium $\varphi_0 \in \mathbb{C}^{n\times n}$ [20, Theorem 16].
2) The escape time T_{esc} for the Riccati equation (7) corresponds to the first time $t > 0$ where $t \rightarrow \mathcal{D}(t, \varphi_0)$ vanishes [19].
3) The vanishing time of $t \rightarrow \mathcal{D}(t, \varphi_0)$ does not depend on the equilibrium φ_0 [19, Lemma 2].

Where: $\mathcal{D}(t, \varphi_0) = det[I + \int_0^t e^{\tilde{A}^T p} S e^{\tilde{A} p} dp(M - \varphi_0)]$

$$\tilde{A}(\varphi_0) = A^T - \varphi_0 S$$

The last step is to prove the continuous solution of the Riccati equation (7), if it exists, is unique. Such a result is guaranteed by the Radon lemma (Theorem 3.1.1, [19]).

V. CHARACTERIZATION OF BASINS OF ATTRACTION

In this section, we characterize the mapping between agents' initial positions and their final destination choice. By doing so, we subdivide the initial positions space into regions called basins of attraction. For that, we, first, define the basin of attraction as follows:

$$D_j(\bar{X}) = \{x \in \mathbb{R}^n | J_{j,M}(x, u^*_{d_j,M}, \bar{X}) \leq J_{k,M}(x, u^*_{d_k,M}, \bar{X}), \forall k \in \{1, \dots, l\}\} \tag{13}$$

Assumption 1: Without loss of generality, we assume that if the agent's initial position $x_0 \in \bigcap_{m=1}^k D_{j_m}(\bar{X})$, for some $j_1 < \ldots < j_k$, then the player i goes toward j_1.
Under Assumption 1, the evacuation problem (2) admits a unique optimal control law. The next theorem provides an explicit form for the basins of attractions and its proof is essentially computational.

***Theorem** 5.1:* The basins of attraction are separated by hyperplanes and have the following expression:

$$D_j(\bar{X}) = \{x \in \mathbb{R}^n | \alpha_{jk}^T x \leq \beta_{jk} + \delta_{jk}(\bar{X}), \forall k \in \{1, \dots, l\}\} \tag{14}$$

where:

$$\alpha_{j,k} = \gamma_{d_j}(0) - \gamma_{d_k}(0), \beta_{j,k} = d_k^T R_d d_k - d_j^T R_d d_j$$
$$+ \int_T^0 \frac{1}{4}(\gamma_{d_k}^T S \gamma_{d_k} - \gamma_{d_j}^T S \gamma_{d_j}) + d_j^T R_d d_j - d_k^T R_d d_k ds$$
$$\delta_{j,k}(\bar{X}) = \frac{1}{2}\int_T^0 (\gamma_{d_k}^T S \zeta_{\bar{X}} - \gamma_{d_j}^T S \zeta_{\bar{X}}) ds$$
$$\gamma_{d_j}(t) = -2\phi^-(t,T) M d_j + 2\int_T^t \phi^-(t,s) ds R_d d_j$$
$$\zeta_{\bar{X}}(t) = -2\int_T^t \phi^-(t,s) R_x \bar{X}(s) ds$$
$$\phi^-(t,x) = \exp\left(\int_x^t \varphi^T(s) ds S - A^T(t-x)\right)$$

VI. FIXED POINT AND NASH EQUILIBRIUM

In this section, we will prove the existence of an $\epsilon-$Nash equilibrium for the proposed evacuation model in the multidimensional case. We start by establishing the MFG equations. Since each agent is heading toward an exit $d_j, j \in \{1..l\}$ and by using the linearity of both agents dynamics and ψ_{d_j}, the system of MFG equations can be expressed as follows:

$$\begin{aligned} &\frac{d}{dt}\bar{X} = (A - S\varphi(t))\bar{X} - \frac{1}{2}S\psi_{d_\lambda}(t) \\ &\frac{d}{dt}\psi_{d_\lambda} = (\varphi^T S - A^T)\psi_{d_\lambda} + 2(R_d d_\lambda - R_x \bar{X}) \\ &\text{with } \bar{X}(0) = \bar{X}_0,\ \psi_{d_\lambda}(T) = -2M d_\lambda \end{aligned} \tag{15}$$

where $d_\lambda = \mathbb{E}_{i\in\{1..l\}}(d_i) = \sum_{i=1}^l p_i d_i$ is a convex combination of exit positions $\{d_i, i \in \{1..l\}\}$ and $\{p_1, .., p_l\}$ are the exit choices probability distribution.
Now, we explicit the form of the MFG fixed point. This allows us to transform our search for the MFG fixed point from one in the space of functions into one in a probability simplex of size l. The following theorem is a generalization of Theorem 3 and 8 in [17].
Assumption 2: We assume that the probability measure $\mathcal{P}_0$ is such that the measure of hyperplanes is zero.

***Theorem** 6.1:* Under Assumption 2, the following three statements hold:

1) If $\bar{X}$ is a fixed point for the system of MFG equations (15), then, it can be written as:
$$\bar{X}(t) = \mathcal{F}_1(t)\bar{X}(0) + \mathcal{F}_2(t) d_\lambda$$
where $\mathcal{F}_1, \mathcal{F}_2$ are some continuous functions of t.
2) The system of MFG equations (15) admits at least one fixed point.
3) The decentralized strategies in (6) lead to an $\epsilon-$Nash equilibrium.

Proof: Due to space constraints, we only provide the main idea for the proof of statement (1). Any solution for (15) corresponds to some exits choices probability distribution. Therefore, we fix the probability distribution of exits choices $\{p_1...p_l\}$. Then, we define the variable $n(t) = \varphi\bar{X}(t) + \frac{1}{2}\psi_{d_\lambda}(t)$ where $d_\lambda = \sum_{j=1}^{l} p_j d_j$. We recognize that $\bar{X}, n$ correspond respectively to the state and costate of the following optimal control problem:

$$\begin{aligned}\min_u \int_0^T \frac{1}{2}(x-d_P)^T R_d (x-d_P) + \frac{1}{2}u^T R_u u dt \\ + \frac{1}{2}(x-d_P)^T M (x-d_P) \\ \text{s.t. } \dot{x} = Ax + Bu,\ x(0) = \bar{X}(0)\end{aligned}$$

Thus, any solution $\bar{X}$ for (15) can be written as:

$$\bar{X}(t) = \mathcal{F}_1(t)\bar{X}(0) + \mathcal{F}_2(t)d_\lambda$$

where $\mathcal{F}_1, \mathcal{F}_2$ are some continuous functions of t.

For statement (2), we use the proof of Theorem 5 in [17], which relies on Brouwer's fixed point theorem. For statement (3), we consider a finite population with N agents coupled through the cost function defined in (2) and we replace the population's mean by its empirical mean $\bar{X}^{(N)}$. Then, we use the proof of theorem 8 in [17] to establish the statement. ■

VII. Simulations and Numerical results

To illustrate the population behavior, we consider 1000 agents uniformly spread over the domain $\{(X,Y)|X \in [-10,10], Y \in [-10,10]\}$ and moving in $\mathbb{R}^2$ according to dynamics $A = 0_{2\times2}, B = I_{2\times2}$. Exits are placed at:

$$d_1 = \begin{pmatrix}-20.3\\20\end{pmatrix}, d_2 = \begin{pmatrix}20.9\\19.6\end{pmatrix}, d_3 = \begin{pmatrix}-20.2\\-20.2\end{pmatrix}, d_4 = \begin{pmatrix}19.9\\-20.5\end{pmatrix}$$

We vary the matrix weight $R_d \in \{0, 8, 30\} \times I_{2\times2}$ to depict the different psychological attitudes (i.e., non stressed, moderately stressed, and highly stressed). R_x describes the social interactions and is either definite positive for the congestion avoidance case or definite negative for the crowd following case $R_x = \pm \begin{pmatrix}18 & 0\\0 & 9\end{pmatrix}$. The rest of the model parameters are as follows:

$$M = 8.10^3 \times I_{2\times2}, R_u = 2.10^5 \times I_{2\times2}$$

By using Theorem 4.1, we calculate the escape time for the congestion without stress case 104.71 s and the congestion with moderate stress 140 s. The high stress case does not exhibit a finite escape time since $R_x \succ R_d$. Such variation illustrates the stress effect which retains agents from diverging. For our evacuation model, we choose the time horizon to be less than both escape times T=90 s.

To evaluate the agents separating distance, we proceed by calculating the mean position of each group heading to a given destination. Then, we find, at each time t, two circles $\mathcal{C}_{max}(t)$ and $\mathcal{C}_{min}(t)$ centered at the origin where $\mathcal{C}_{max}(t)$ is the smallest circle enclosing all the calculated group mean positions at given instant t and $\mathcal{C}_{min}(t)$ is the biggest circle not containing any of the means positions. We define the groups' separating distance $\Delta(t)$ as the difference between the radius of $\mathcal{C}_{max}(t)$, and $\mathcal{C}_{min}(t)$.

Fig.1(a) and Fig.1(b) represent the mean trajectories and the groups' distances, respectively. Dashed curves represent congestion cases, while continuous curves are for crowd following cases. Blue, orange, and black curves designate respectively situations with low, moderate, and high levels of stress.

The simulation results show that the social interaction effect impacts the groups' distance curve shape, which tends to be convex for the congestion cases and concave for the crowd following cases (See Fig.1(b)). On the other hand, Fig 2 shows that when increasing stress, agents tend to rush toward their chosen exits at the beginning of the evacuation process then slow down at the end. Besides, independently of considered social interaction scenarios, the population density increases with the stress factor. Indeed, the more stressed the agents are, the thinner the groups' trajectories branches are (See Fig 2). We also note that for the stress-free crowd following scenario, no agent chooses exit d_2, and by increasing stress, agents consider it and the exits choices probability distribution approaches 0.25 for all destinations (See Table I). Such tendency illustrates how stress urges agents to choose the closest exit to their initial position and to rush toward it. Finally, except for the stress-free crowd following case where agents stuck together and do not consider exit d_2, the population's mean remains close to its initial value. Such observation is due to the quasi-symmetry of exits around agents' initial space distribution and their initial spatial uniform distribution.

VIII. Conclusion

This work proposes an MFG based model for evacuation processes. It introduces several aspects potentially impacting the effectiveness of a given evacuation plan: social interactions through congestion avoidance or crowd following behavior, as well as agents' psychological state through stress. The proposed model may be used for crowd motion prediction during an evacuation process or as a microscopic based model to describe the individual reasoning mechanism at a decision-making level. We also provide sufficient conditions for the solvability of the model. We provide a method to accelerate numerical simulations by subdividing the destinations choices space by hyperplanes. In future work, we aim at making agents' behavior dependent on the mean of their destination choice cohort rather than the global population mean. Furthermore, we intend to introduce a collaborative aspect for the current model by allowing agents' control over the population variance.

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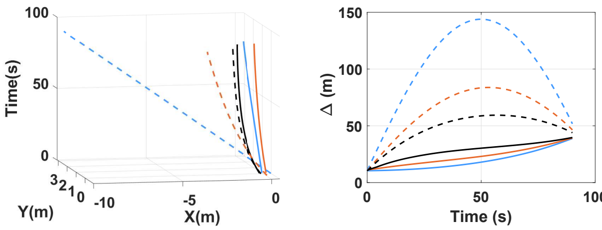
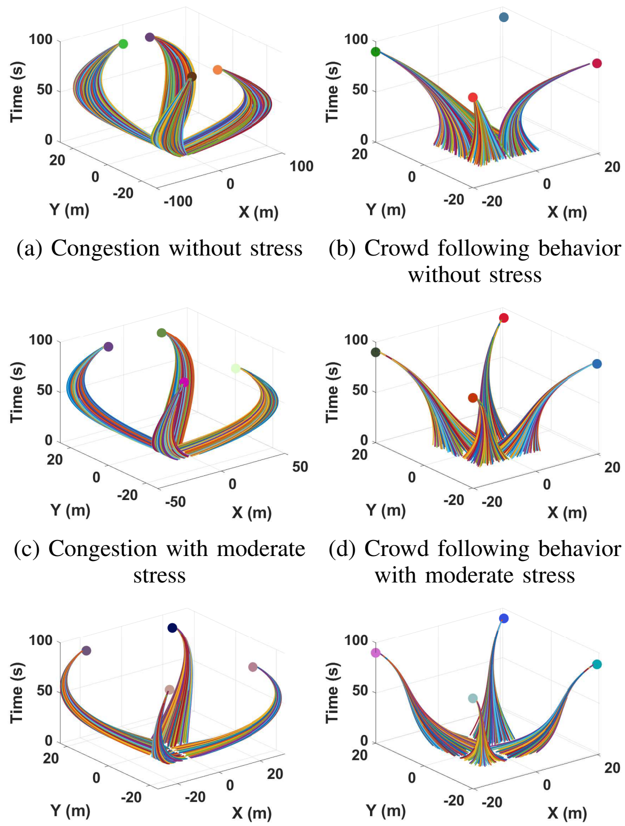


(a) Means trajectories (b) Groups' distances

Fig. 1: Means and groups' distances



(a) Congestion without stress (b) Crowd following behavior without stress

(c) Congestion with moderate stress (d) Crowd following behavior with moderate stress

(e) Congestion with high stress (f) Crowd following behavior with high stress

Fig. 2: Agents trajectories for given stress levels and social interaction scenarios

TABLE I: Destinations choices probabilities

		d1	d2	d3	d4
Congestion avoidance	**without stress**	0.297	0.228	0.224	0.251
	moderate stress	0.276	0.236	0.236	0.252
	high stress	0.302	0208	0.234	0.256
Crowd following behavior	**without stress**	0.584	0	0.155	0.261
	moderate stress	0.345	0.137	0.245	0.273
	high stress	0.294	0.208	0.252	0.246

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