Chapter 3

Sampling and Sampled-Data Systems

3.1 Introduction

Today, virtually all control systems are implemented digitally, on platforms that range from large computer systems (e.g. the mainframe of an industrial SCADA[1] system) to small embedded processors. Since computers work with digital signals and the systems they control often live in the analog world, part of the engineered system must be responsible for converting the signals from one domain to another. In particular, the control engineer should understand the principles of sampling and quantization, and the basics of Analog to Digital and Digital to Analog Converters (ADC and DAC).

Control systems that combine an analog part with some digital components are traditionally referred to as *sampled-data systems*. Alternative names such as hybrid systems or cyber-physical systems (CPS) have also been used more recently. In this case, the implied convention seems to be that the digital part of the system is more complex than in traditional sampled-data systems, involving in particular logic statements so that the system can switch between different behaviors. Another important concern for the development of CPS is system integration, since we must often assemble complex systems from heterogeneous components that might be designed independently using perhaps different modeling abstractions. In this chapter however we are concerned with the classical theory of sampled-data system, and with digital systems that are assumed to be essentially dedicated to the control task, and as powerful for this purpose as needed. Even in this case, interesting questions are raised by the capabilities of modern digital systems, such as the possibility of very high sampling rates. Moreover, we can build on this theory to relax its unrealistic assumptions for modern embedded implementation platforms and to consider more complex hybrid system and CPS issues, as discussed in later chapters.

[1]SCADA = Supervisory Control and Data Acquisition

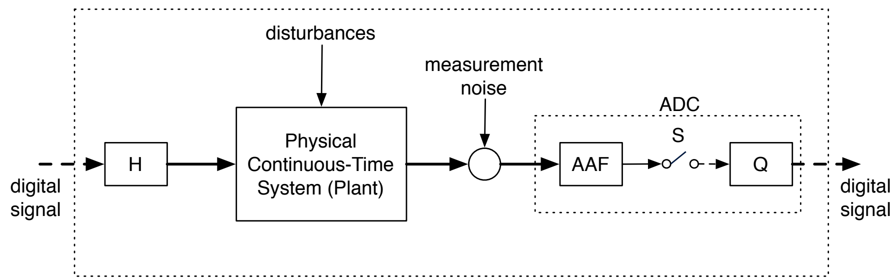


Figure 3.1: Sampled-data model. H = hold, S = sampler, Q = quantizer (+ decoder), AAF = anti-aliasing (low-pass) filter. Continuous-time signals are represented with full lines, discrete-time signals with dashed lines.

Organization of Sampled-Data Systems

The input signals of a digital controller consist of discrete sequences of finite precision numbers. We call such a sequence a digital signal. Often we ignore quantization (i.e., finite precision) issues and still call the discrete sequence a digital signal. In sampled-data systems, the plant to be controlled is an analog system (continuous-time, and usually continuous-state), and measurements about the state of this plant that are initially in the analog domain need to be converted to digital signals. This conversion process from analog to digital signals is generally called sampling, although sampling can also refer to a particular part of this process, as we discuss below. Similarly, the digital controller produces digital signals, which need to be transformed to analog signals to actuate the plant. In control systems, this transformation is typically done by a form of signal holding device, most commonly a zero-order hold (ZOH) producing piecewise constant signals, as discussed in section 3.3. Fig. 3.1 shows a sampled-data model, i.e. the continuous plant together with the DAC and ADC devices, which takes digital input signals and produces digital output signals and can be connected directly to a digital controller. The convention used throughout these notes is that continuous-time signals are represented with full lines and sampled or digital signals are represented with dashed lines. Note that the DAC and ADC can be integrated for example on the microcontroller where the digital controller is implemented, and so the diagram does not necessarily represents the spatial configuration of the system. We will revisit this point later as we discuss more complex models including latency and communication networks. The various parts of the system represented on Fig. 3.1 are discussed in more detail in this chapter.

3.2 Sampling

Preliminaries

We first introduce our notation for basic transforms, without discussing issues of convergence. Distributions (e.g. Dirac delta) are also used informally (Fourier transforms can be defined for tempered distributions).

Continuous-Time Fourier Transform (CTFT): For a continuous-time function $f(t)$, its Fourier transform is

$$\hat{f}(\omega) = \mathcal{F}f(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt.$$

Inverting the Fourier transform, we then have

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)e^{i\omega t}d\omega.$$

Laplace Transform: generalizes the Fourier transform. In control however, one generally uses the one-sided Laplace transform

$$\hat{f}(s) = \mathcal{L}f(s) = \int_{0^-}^{\infty} f(t)e^{-st}dt, \quad s \in \mathbb{C},$$

which is typically not an issue since we also assume that signals are zero for negative time. We can invert it using

$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \hat{f}(s)e^{st}ds,$$

where c is a real constant that is greater than the real parts of all the singularities of $\hat{f}(s)$.

Discrete-Time Fourier Transform (DTFT): for a discrete-time sequence $\{x[k]\}_k$, its Fourier transform is

$$\hat{x}(e^{i\omega}) = \mathcal{F}x(e^{i\omega}) = \sum_{k=-\infty}^{\infty} x[k]e^{-i\omega k}.$$

The inversion is

$$x[k] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{x}(e^{i\omega})e^{i\omega k}d\omega,$$

where the integration could have been performed over any interval of length 2π. Since we use both continuous-time and discrete-time signals, we use the term discrete-time Fourier transform for the Fourier transform of a discrete-time sequence. The notation should remind the reader that $\hat{x}(e^{i\omega})$ is periodic of period 2π (similarly in the following, we use $\hat{x}(e^{i\omega h})$, which is periodic of period $\omega_s = 2\pi/h$). It is sufficient to consider the DTFT of a sequence over the interval $(-\pi, \pi]$. The frequencies close to $0 + 2k\pi$ correspond to the low frequencies, and the frequencies close to $\pi + 2k\pi$ to the high frequencies. The theory of the DTFT is related to that of the Fourier *series* for the periodic function $\hat{x}(e^{i\omega})$.

z-transform and λ-transform: generalizing the DTFT for a discrete sequence $\{x[k]\}_k$, we have the two-side z-transform

$$\hat{x}(z) = \sum_{k=-\infty}^{\infty} x[k] z^{-k}, \quad z \in \mathbb{C}.$$

In control however, we often use the one-sided z-transform

$$\hat{x}(z) = \sum_{k=0}^{\infty} x[k] z^{-k}, \quad z \in \mathbb{C},$$

but this not an issue, because the sequences are typically assumed to be zero for negative values of k. The z-transform is analogous to the Laplace transform, now for discrete-time sequences. It is often convenient to use the variable $\lambda = 1/z$ instead, and we call the resulting transform the λ-transform

$$\hat{x}(\lambda) = \sum_{k=-\infty}^{\infty} x[k] \lambda^{k}, \quad \lambda \in \mathbb{C}.$$

For G a DT LTI system (discrete-time linear time-invariant system) with matrices A, B, C, D, as usual and impulse response $\{g(k)\}$, its transfer function (a matrix in general) is

$$g(\lambda) = D + \lambda C(I - \lambda A)^{-1} B,$$

or

$$g(z) = D + C(zI - A)^{-1} B,$$

and also denoted

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right].$$

Periodic Sampling of Continuous-Time Signals

In sampled-data systems the plant lives in the analog world and data conversion devices must be used to convert its analog signals in a digital form that can be processed by a computer. We first consider the output signals of the plant, which are transformed into digital signals by an Analog-to-Digital Converter (ADC). For our purposes, an ADC consists of four blocks[2].

[2]this discussion assumes a "Nyquist-rate" ADC rather than an "oversampling" Delta-Sigma converter, see [Raz95].

1. First, an analog low-pass filter limits the signal bandwidth so that subsequent sampling does not alias unwanted noise or signal components into the signal band of interest. In a control system, the role of this *Anti-Aliasing Filter* (AAF) is also to remove undesirable high-frequency disturbances that can perturb the behavior of the closed-loop system. Aliasing is discussed in more details in the next paragraph. For the purpose of analysis, the AAF can be considered as part of the plant (the dynamics of the AAF can in some rare instances be neglected, see [ÅW97, p.255]). Note that most analog sensors include some kind of filter, but the filter is generally not chosen for a particular control application and therefore might not be adequate for our purposes.

2. Next, the filter output is sampled to produce a discrete-time signal, still real-valued.

3. The amplitude of this signal is then quantized, i.e., approximated by one of a fixed set of reference levels, producing a discrete-time discrete-valued signal.

4. Finally, a digital representation is of this signal is produced by a decoder and constitutes the input of the processor. From the mathematical point of view, we can ignore this last step which is simply a choice of digital signal representation, and work with the signal produced by the quantizer.

Let $x(t)$ be a continuous-time signal. A sampler (block S on Fig. 3.1) operating at times t_k with $k = 0, 1, \ldots$ or $k = \ldots, -1, 0, 1, \ldots$, takes x as input-signal and produces the discrete sequence $\{x_k\}_k$, with $x_k = x(t_k)$. Traditionally, sampling is performed at regular intervals, as determined by the *sampling period* denoted h, so that we have $t_k = kh$. This is the situation considered in this chapter. We then let $\omega_s = \frac{2\pi}{h}$ denote the sampling frequency (in rad/s), and $\omega_N = \frac{\omega_s}{2} = \frac{\pi}{h}$ is called the Nyquist frequency. Note that we will revisit the periodic sampling assumption in later chapters, because it can be hard to satisfy in networked embedded systems.

Aliasing

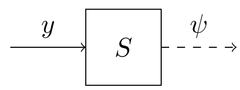
Sampling is a linear operation, but sampled systems in general are *not* time-invariant. Perhaps more precisely, consider a simple system HS, consisting of a sampler followed by a perfectly synchronized ZOH device. This system maps a continuous-time signal into another one, as follows. If u is the input signal, then the output signal y is

$$y(t) = u(t_k), \;\; t_k \leq t < t_{k+1}, \forall k,$$

where $\{t_k\}_k$ is the sequence of sampling (and hold) times. It is easy to see that it is linear and causal, but not time-invariant. For example, consider the output produced when the input is a simple ramp, and shift the input ramp in time. This system is *periodic* of period h if the sampling is periodic with this period, in the sense that shifting the input signal u by h results in shifting the output y by h. Indeed, periodically sampled systems are often periodic systems.

Exercise 2. Assume that two systems $H_1 S_1$ and $H_2 S_2$ with sampling periods h_1 and h_2 are connected in parallel. For what values of h_1, h_2 is the connected system periodic?

In particular, sampled systems do not have a transfer function, and new frequencies are created in the signal by the process of sampling, leading to the distortion phenomenon known as *aliasing*. Consider a periodic sampling block S, with sampling period h, analog input signal y and discrete output sequence ψ, i.e., $\psi[k] = y(kh)$ for all k.



The following result relates the Fourier transforms of the continuous-time input signal and the discrete-time output signal. Define the periodic-extension of $\hat{y}(\omega)$ by

$$\hat{y}_e(\omega) := \sum_{k=-\infty}^{\infty} \hat{y}(\omega + k\omega_s),$$

and note that $\hat{y}_e$ is periodic of period ω_s, and is characterized by its values in the band $(-\omega_N, \omega_N]$.

Lemma 3.2.1. *The DTFT of $\psi = \{y(kh)\}_k$ and the CTFT of y are related by the relation*

$$\hat{\psi}(e^{i\omega h}) = \frac{1}{h}\hat{y}_e(\omega),$$

i.e.,

$$\hat{\psi}(e^{i\omega}) = \frac{1}{h}\hat{y}_e\left(\frac{\omega}{h}\right).$$

Proof. Consider the impulse-train $\sum_k \delta(t - kh)$ (or Dirac comb), which is a continuous-time signal of period h. The Poisson summation formula gives the identity

$$\sum_k \delta(t - kh) = \frac{1}{h}\sum_k e^{ik\omega_s t}.$$

Then define the impulse-train modulation

$$v(t) = y(t)\sum_k \delta(t - kh) = \frac{1}{h}\sum_k y(t)e^{ik\omega_s t}.$$

Taking Fourier transforms, we get

$$\hat{v}(\omega) = \frac{1}{h}\sum_k y(\omega + k\omega_s) = \frac{1}{h}\hat{y}_e(\omega).$$

On the other hand, we can also write

$$v(t) = \sum_k \psi(k)\delta(t - kh).$$

And taking again Fourier transforms

$$\begin{aligned}\hat{v}(\omega) &= \int \left[\sum_k \psi(k)\delta(t - kh)\right] e^{i\omega t} dt \\ &= \sum_k \psi(k) e^{-i\omega kh} \\ &= \hat{\psi}(e^{i\omega h}).\end{aligned}$$

□

In other words, periodic sampling results essentially in the periodization of the Fourier transform of the original signal y (the rescaling by h in frequency is not important here: it is just due to the fact that the discrete-time sequence is always normalized, with intersample distance equal to 1 instead of h for the impulse train). If the frequency content of y extends beyond the Nyquist frequency ω_N, i.e. $\hat{y}$ is not zero outside of the band $(-\omega_N, \omega_N)$, then the sum defining $\hat{y}_e$ involves more than one term in general at a particular frequency, and the signal y is distorted by the sampling process. On the other hand, if $\hat{y}$ is limited to $(-\omega_N, \omega_N)$, then we have $\hat{\psi}(e^{i\omega h}) = y(\omega)$ for the defining interval $(-\omega_N, \omega_N)$ and the sampling block does not distort the signal, but acts as a simple multiplicative block with gain $1/h$. The presence of an anti-aliasing low-pass filter, with *cut-off frequency* at ω_N, is then a means to avoid the folding of high signal frequencies of the continuous-time signals into the frequency band of interest due to the sampling process.

Sampling Noisy Signals

The previous discussion concerns purely deterministic signals and justifies the AAF to avoid undesired folding of high-frequency components in the frequency band of interest. Now for a stochastic input signal y to the sampler, it is perhaps more intuitively clear that direct sampling is not a robust scheme, as the values of the samples become overly sensitive to high-frequency noise. Instead, a popular way of sampling a continuous-time stochastic signal is to integrate (or average) the signal over a sampling period before sampling

$$\psi[k] = \int_{t=(k-1)h}^{kh} y(\tau) d\tau,$$

which is in fact also form of (low-pass) prefiltering since we can rewrite

$$\psi[k] = (f * y)(t)|_{t=kh},$$

where

$$f(t) = \begin{cases} 1, & 0 \leq t < h \\ 0, & \text{otherwise.} \end{cases} \tag{3.1}$$

This filter can be called an averaging filter or "integrate and reset" filter. In other words, the continuous-time signal $y(t)$ first passes through the filter that produces a signal $\bar{y}(t)$, in this case

$$\bar{y}(t) = \int_{(k-1)h}^{t} y(s)ds, \quad \text{for } (k-1)h \leq t < kh.$$

Note that $\bar{y}((kh)^+) = 0, \forall k$, i.e., the filter resets the signal just after the sampling time. Also, even though the signal is reset, if we have access to the samples $\{\psi[k]\}_k$, we can immediately reconstruct the integrated signal $z(t) = \int_0^t y(s)ds$ at the sampling instants, since $z(kh) = \sum_{i=1}^{k} \psi[k]$, and hence the signal $y(t)$ at the sampling instants as well (since $y = \dot{z}$). In Section 3.4 we provide additional mathematical justification for the integrate and reset filter to sample stochastic differential equations.

Exercise 3. Compute the Fourier transform of f in (3.1) and explain why this is a form of low pass filtering.

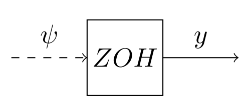
3.3 Reconstruction of Analog Signals: Digital-to-Analog Conversion

Zero-Order Hold

The simplest and most popular way of reconstructing a continuous-time signal from a discrete-time signal in control systems is to simply hold the signal constant until a new sample becomes available. This transforms the discrete-time sequence into a piecewise constant continuous-time signal[3], and the device performing this transformation is called a zero-order hold (ZOH). Let us consider the effect of the ZOH in the frequency domain, assuming an inter-sample interval of h seconds. Introduce

$$r(t) = \begin{cases} 1/h, & 0 \leq t < h \\ 0, & \text{elsewhere.} \end{cases}$$

Then the relationship between the input and output of the zero-order hold device



[3]This is again a mathematical idealization. In practice, the physical output of the hold device is continous.

can be written

$$y(t) = h \sum_k \psi[k]\, r(t - kh).$$

Taking Fourier transforms, we get

$$\begin{aligned} \hat{y}(\omega) &= h \sum_k \psi[k]\, \hat{r}(\omega) e^{-i\omega kh} \\ &= h \hat{r}(\omega) \hat{\psi}(e^{i\omega h}). \end{aligned}$$

Note that we can write

$$r(t) = \frac{1}{h} 1(t) - \frac{1}{h} 1(t - h),$$

and so we have the Laplace transform

$$\hat{r}(s) = \frac{1 - e^{-sh}}{sh}.$$

We then get the Fourier transform

$$\begin{aligned} \hat{r}(\omega) &= \frac{1 - e^{-i\omega h}}{i\omega h} \\ &= e^{-i\omega h/2} \frac{\sin \omega \frac{h}{2}}{\omega \frac{h}{2}}. \end{aligned}$$

Note in particular that $\hat{r}(s) \approx e^{-sh/2}$ at low frequency, and so in this regime $\hat{r}$ acts like a time delay of $h/2$.

Lemma 3.3.1. *The Fourier transforms of the input and output signals of the zero-order hold device are related by the equation*

$$\hat{y}(\omega) = h \hat{r}(\omega) \hat{\psi}(e^{i\omega h}).$$

First-Order Hold

Other types of hold devices can be considered. In particular, we can try to obtain smaller reconstruction errors by extrapolation with high-order polynomials. For example, a first-order hold is given by

$$y(t) = \psi[k] + \frac{t - t_k}{t_k - t_{k+1}} (\psi[k] - \psi[k-1]), \quad t_k \leq t < t_{k+1},$$

where the times t_k are the times at which new samples $\psi[k]$ become available. This produces a piecewise linear CT signal, by continuing a line passing through the two most recent samples. The reconstruction is still discontinuous however. Postsampling filters at the DAC can be used if these discontinuities are a problem.

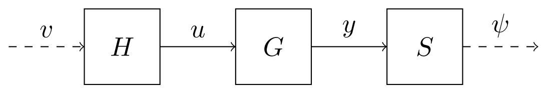


Figure 3.2: Step-invariant transformation for discretizing G. H here is a ZOH.

One can also try a predictive first-order hold

$$y(t) = \psi[k] + \frac{t - t_k}{t_{k+1} - t_k}(\psi[k+1] - \psi[k]), \;\; t_k \leq t < t_{k+1},$$

but this scheme is non-causal since it requires $\psi[k+1]$ to be available at t_k. For implementing a predictive FOH, one can thus either introduce a delay or better use a prediction of $\psi[k+1]$. See [ÅW97, chapter 7] for more details on the implementation of predictive FOH. In any case, an important issue with any type of higher-order hold is that it is usually not available in hardware, largely dominated by ZOH DAC devices[4].

3.4 Discretization of Continuous-Time Plants

Step-Invariant discretization of linear systems

We can now consider the discrete-time system obtained by putting a zero-order hold device H and sampling block S at the input and output respectively of a continuous-time plant G, see Fig. 3.2. Let us assume that S is a periodic sampling block and H is a perfectly synchronized with S. The resulting discrete-time system is denoted $G_d = SGH$, and is called the *step-invariant transformation* of G. The name can be explained by the fact that unit steps are left invariant by the transformation, in the sense that

$$G_d S1 = G_d 1_d = SGH1_d = SG1,$$

where 1 and 1_d are the CT and DT unit steps respectively. Using Lemmas 3.2.1 and 3.3.1, we obtain the following result.

Theorem 3.4.1. *The CTFT $\hat{g}(\omega)$ of G and the DTFT of G_d obtained by a step-invariant transformation are related by the equation*

$$\hat{g}_d(e^{i\omega h}) = \sum_{k=-\infty}^{\infty} \hat{g}(\omega + k\omega_s)\hat{r}(\omega + k\omega_s),$$

or

$$\hat{g}_d(e^{i\omega}) = \sum_{k=-\infty}^{\infty} \hat{g}\left(\frac{\omega}{h} + k\omega_s\right)\hat{r}\left(\frac{\omega}{h} + k\omega_s\right).$$

[4]There is also the possibility of adding an inverse-sinc filter and a low-pass filter at the back-end of the ZOH DAC, see [Raz95].

Proof. To compute $\hat{g}_d$, we take v on Fig. 3.2 to be the discrete-time unit impulse, whose DTFT is the constant unit function. Then we have $\hat{u}(\omega) = h\hat{r}(\omega)$, thus $y = \hat{g}(\omega)\hat{u}(\omega)$, and the result follows from Lemma 3.2.1. □

Note that if the transfer function $\hat{g}(\omega)$ of the continous-time plant G that is bandlimited to the interval $(-\omega_N, \omega_N)$, we have then

$$\hat{g}_d(e^{i\omega h}) = \hat{g}(\omega)\hat{r}(i\omega), \;\; -\omega_N < \omega < \omega_N,$$

and at low frequencies

$$\hat{g}_d(e^{i\omega h}) \approx \hat{g}(i\omega),$$

or somewhat more precisely, the ZOH essentially introduces a delay of $h/2$. Otherwise distortion by aliasing occurs, which can significantly change the frequency content of the system. This in particular intuitively justifies the presence of an anti-aliasing filter (AAF) with cutoff frequency $\omega_N < \omega_s/2$ at the output of the plant, if a digital control implementation with sampling frequency ω_s is expected. Indeed, aliasing can potentially introduce undesired oscillations impacting the performance of a closed-loop control system without proper AAF. The following example, taken from [ÅW97, chapter 1], illustrates this issue.

Example 3.4.1. Consider the continuous-time control system shown on Fig. 3.3. The plant in this case is a disk-drive arm, which can be modeled approximately by the transfer function

$$P(s) = \frac{k}{Js^2},$$

where k is a constant and J is the moment of inertia of the arm assembly. The arm is should to be positioned with great precision at a given position, in the fastest possible way in order to reduce access access time. The controller is implemented for this purpose, and we focus here to the response to step inputs u_c. Classical continuous-time design techniques suggest a controller of the form

$$K(s) = M\left(\frac{b}{a}U_c(s) - \frac{s+b}{s+a}Y(s)\right) = M\left(\frac{b}{a}U_c(s) - Y(s) + \frac{a-b}{s+a}Y(s)\right), \tag{3.2}$$

where

$$a = 2\omega_0, \; b = \omega_0/2, \text{ and } M = 2\frac{J\omega_0^2}{k},$$

and ω_0 is a design coefficient. The transfer function of the closed loop system is

$$\frac{Y(s)}{U(s)} = \frac{\frac{\omega_0^2}{2}(s+2\omega_0)}{s^3 + 2\omega_0 s^2 + 2\omega_0^2 s + \omega_0^3}. \tag{3.3}$$

This system has a settling time to 5% equal to $5.52/\omega_0$. The step response of the closed-loop system is shown on Fig. 3.3 as well.

Let us now assume that a sinusoidal signal $n(t) = 0.1\sin(12t)$ of amplitude 0.1 and frequency 12 rad/s perturbs the measurements of the plant output. It turns out that this disturbance has little impact on the performance of the continuous-time closed-loop system, see the left column of Fig. 3.4. Then consider the following simple discretization of the controller (3.2). First, a continuous-time state-space realization of the transfer function $K(s)$ is realized by

$$
\begin{aligned}
u(t) &= M\left(\frac{b}{a}u_c(t) - y(t) + x(t)\right) \\
\dot{x}(t) &= -ax + (a-b)y.
\end{aligned}
$$

Next, we approximate the derivative of the controller state with a simple forward difference scheme (we will see soon that we could be more precise here, but this will do for now)

$$
\frac{x(t+h) - x(t)}{h} = -ax(t) + (a-b)y(t),
$$

where h is the sampling period. We then obtain the following approximation of the continuous-time controller

$$
\begin{aligned}
x[k+1] &= (1-ah)x[k] + h(a-b)y[k], \\
u[k] &= M\left(\frac{b}{a}u_c[k] - y[k] + x[k]\right),
\end{aligned}
$$

where $u_c[k] = u_c(kh)$ and $y[k] = y(kh)$ are the sampled values of the input reference and (noise perturbed) plant output. A Simulink model of this implementation is shown on Fig. 3.5. As you can see on the right of Fig. 3.4, for a choice of sampling period $h = 0.5$ that is too large, there is a significant deterioration of the output signal due to the presence of a clearly visible low frequency component.

This phenomenon is explained by the aliasing phenomenon, as discussed above. We have $\omega_s = 2\pi/h = 2\pi/0.5 \approx 12.57$ rad/s, and the measured signal has a frequency 12 rad/s, well above the Nyquist frequency. After sampling, we then have the creation of a low frequency component with the frequency $12.57 - \omega_s = 0.57$ rad/s, in other word with a period of approximately 11 s, which is the signal observed on the right of Fig. 3.4.

Exercise 4. Derive (3.3).

Step-Invariant discretization of linear state-space systems

In this section, we consider a state-space realization of an LTI plant instead of working in the frequency domain as in the previous paragraph. It turns out that the step-invariant discretization of such as plant can be described *exactly* (i.e., without approximation error) by a discrete-time LTI system. This

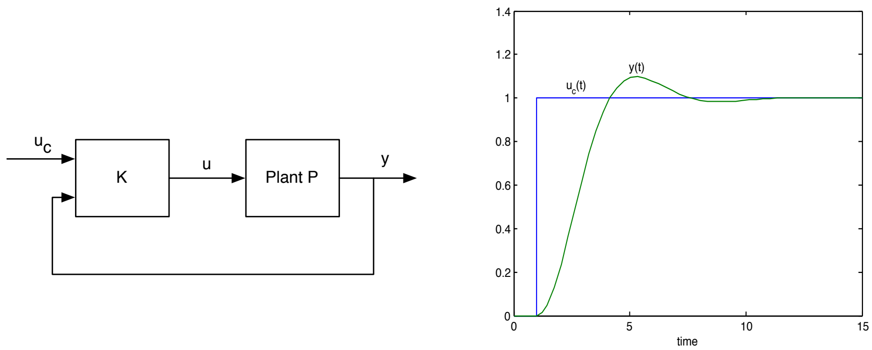


Figure 3.3: Feedback control system and step response for example 3.4.1 with $J = k = \omega_0 = 1$.

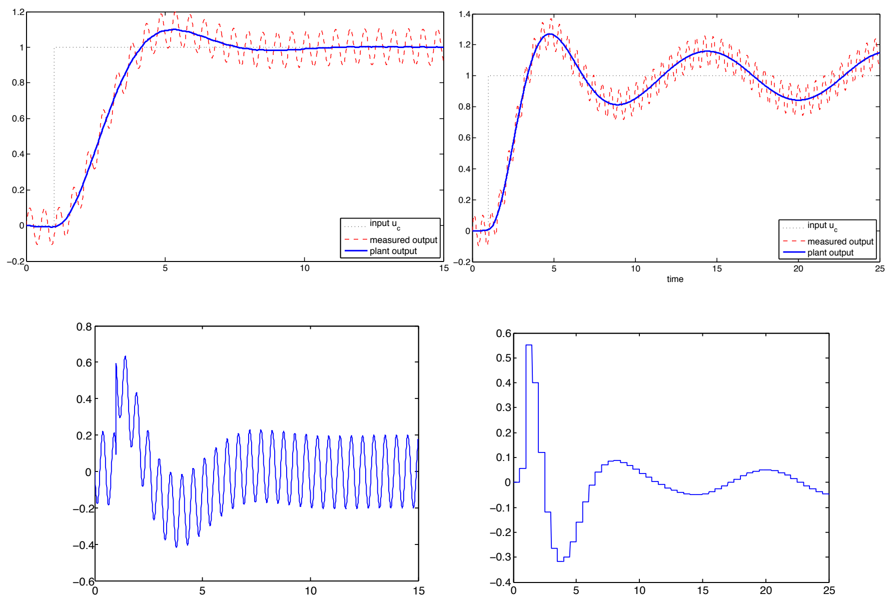


Figure 3.4: Effect of a periodic perturbation on the continuous-time design (left) and discretized design with sampling period 0.5 (right), for example 3.4.1. The bottom row shows the analog input u to the plant. The analog controller provides a significant action, whereas the digital controller does not seem to be able to detect the high-frequency measurement noise.

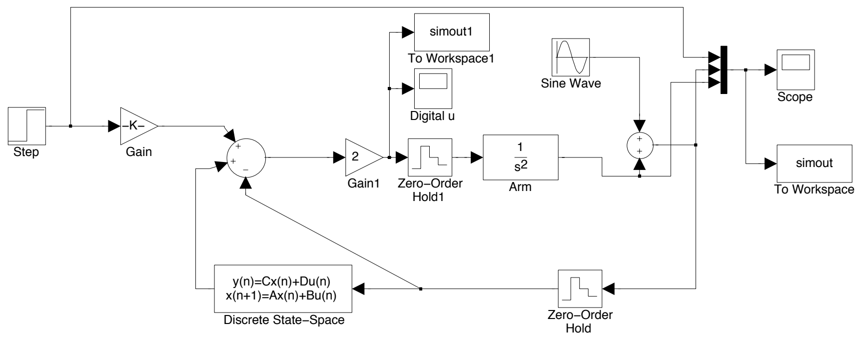


Figure 3.5: Simulink model for the digital implementation of example 3.4.1.

discrete-time system gives exactly the state of the continuous-time plant at the sampling instants. Although this process might not be able to detect certain hidden oscillations in the continuous-time system, it forms the basis of a popular design approach that consists in working only with the discretized version of the plant and using discrete-time control methods.

Hence suppose that we have a CT LTI system G with state-space realization

$$\dot{x} = Ax + Bu \tag{3.4}$$

$$y = Cx + Du, \tag{3.5}$$

and consider the step-invariant transformation SGH, with sampling times $\{t_k\}_k$. The control input u to the CT plant in (3.4), (3.5) is then piecewise constant equal to $u[k]$ on $t_k \leq t < t_{k+1}$. We are also interested in the sampled values of the output $y[k] = y(t_k)$. Directly integrating the differential equation (3.4), we have

$$x(t) = e^{A(t-t_k)}x(t_k) + \int_{t_k}^{t} e^{A(t-\tau)}Bd\tau \; u[k], \text{ for } t_k \leq t < t_{k+1}.$$

In particular, for a periodic sampling scheme with $t_{k+1} - t_k = h$,

$$x(t_{k+1}) = e^{Ah}x(t_k) + \int_{t_k}^{t_{k+1}} e^{A(t_{k+1}-\tau)}Bd\tau \; u[k]$$

$$x(t_{k+1}) = e^{Ah}x(t_k) + \int_{0}^{h} e^{A\tau}Bd\tau \; u[k], \text{ for } t_k \leq t < t_{k+1}.$$

Writing $x[k] = x(t_k)$ for all k, the state of the plant at the sampling times can then be described by the following LTI difference equation

$$x[k+1] = A_d x[k] + B_d u[k],$$

with $A_d = e^{Ah}$ and $B_d = \int_0^h e^{A\tau} B d\tau$. Note that this exact discretization could have been used in example 3.4.1 instead of the approximate Euler scheme. In summary, with *periodic sampling* the plant is represented at the sampling instants by the DT LTI system

$$\begin{aligned} x[k+1] &= A_d x[k] + B_d u[k] \\ y[k] &= C x[k] + D u[k]. \end{aligned}$$

Remark. The matrix exponential, necessary to compute A_d, can be computed using the MATLAB `expm` funtion. There are various ways of computing B_d. The most straightforward is to simply use the MATLAB function `c2d` which performs the step-invariant discretization with a sampling period value provided by the user. Actually this function also provides other types of discretization, including plant discretization using FOH, and other types of discretization for continuous-time controllers (rather than controlled plants), discussed later in this chapter.

A few identities are sometimes useful for computations or in proofs. First define

$$\Psi = \int_0^h e^{A\tau} d\tau = Ih + \frac{Ah^2}{2!} + \dots$$

Then we have $B_d = \Psi B$, and $A_d = I + A\Psi$. Moreover, we have the following result.

Lemma 3.4.2. *Let A_{11} and A_{22} be square matrices, and define for $t \geq 0$*

$$\exp\left\{ t \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \right\} = \begin{bmatrix} F_{11}(t) & F_{12}(t) \\ 0 & F_{22}(t) \end{bmatrix}. \tag{3.6}$$

Then $F_{11}(t) = e^{tA_{11}}$, $F_{22} = e^{tA_{22}}$, and

$$F_{12}(t) = \int_0^t e^{(t-\tau)A_{11}} A_{12} e^{\tau A_{22}} d\tau.$$

Using this lemma, one can compute A_d and B_d using only the matrix exponential function for example, by taking $t = h, A_{11} = A, A_{22} = 0, A_{12} = B$, so that $F_{11}(h) = A_d$ and $F_{12}(h) = B_d$.

Proof. The expressions for F_{11} and F_{22} are immediate since the matrices are block triangular. To obtain F_{12}, differentiate (3.6)

$$\frac{d}{dt} \begin{bmatrix} F_{11}(t) & F_{12}(t) \\ 0 & F_{22}(t) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} F_{11}(t) & F_{12}(t) \\ 0 & F_{22}(t) \end{bmatrix},$$

hence

$$\frac{d}{dt} F_{12}(t) = A_{11} F_{12}(t) + A_{12} F_{22}(t).$$

We then solve this differential equation, using the facts $F_{22}(t) = e^{tA_{22}}$ and $F_{12}(0) = 0$. □

Discretization of Linear Stochastic Differential Equations

Mathematically, it is also impossible to sample directly a signal containing white noise, a popular form of disturbance in control and signal processing models. Formally, the autocovariance function of a continuous-time zero-mean, vector valued white noise process $w(t)$ is a Dirac delta

$$E[w(t)w(t')^T] = r(t-t') = W\delta(t-t'). \tag{3.7}$$

In other words, the values of the signal at different times are uncorrelated, as for discrete-time white noise. The power spectral density of w is defined as the Fourier transform of the autocovariance function

$$\phi_w(\omega) = \int_{-\infty}^{\infty} r(t)e^{-i\omega t}dt = \int_{-\infty}^{\infty} W\delta(t)e^{-i\omega t}dt = W,$$

hence W is called the *power spectral density matrix* of w. The frequency content of w is flat with infinite bandwidth. Hence intuitively, according to the frequency folding phenomenon illustrated in lemma 3.2.1, the resulting sampled signal would have infinite power in the finite band $(-\omega_N, \omega_N)$.

Mathematically rigorous theories for manipulating models involving white noise are usually developed by working instead with an integral version of white noise

$$B(t) = \int_0^t w(s)ds, \quad B(0) = 0.$$

One can then use this theory to justify a posteriori the engineering formulas often formulated in terms of Dirac deltas such as (3.7) [5]. The stochastic process B has zero mean value and its increments $I(s,t) = B(t) - B(s)$ over disjoint intervals are uncorrelated with covariance

$$\mathbb{E}[(B(t)-B(s))(B(t)-B(s))^T] = |t-s|W.$$

For this reason, $W dt$ is sometimes called the incremental covariance matrix of the process B. The stochastic process B is called Brownian motion or Wiener process if in addition the increments have a Gaussian distribution. Note that in this case the increments over disjoint intervals are in fact independent, as a consequence of a well-known property of Gaussian random variables. We then called the corresponding white noise (zero-mean) *Gaussian white noise*.

Gaussian white noise (or equivalently the Wiener process) is the most important stochastic processes for control system applications, in particular because one can derive from it other noise processes with a desired spectral density by using it as an input to an appropriate filter (this is a consequence of spectral factorization, which you might want to review). The advantage of this approach is that we can then only work with white noise and take advantage of the uncorrelated samples property to simplify computations.

[5] Another approach would be to work with white noise rigorously using the theory of distributions, but in general this is unnecessarily complicated and it is just simpler to use integrated signals.

Continuous-time processes with stochastic disturbances are thus often described by a stochastic differential equation, e.g. of linear form

$$\frac{dx}{dt} = Ax + Bu + w(t),$$

where w is a zero-mean white noise process with power spectral density matrix W, which moreover we will assume to be Gaussian. Mathematically, it is then more rigorous to write this equation in the incremental form

$$dx = (Ax + Bu)dt + dB_1(t), \tag{3.8}$$

where B_1 is a Wiener process with incremental covariance $R_1 dt$. This equation just means the integral form

$$x(t) = x(0) + \int_0^t (Ax + Bu)dt + \int_0^t dB_1(t),$$

where the last term is called a stochastic integral and can be rigorously defined. Similarly, white noise can be used to model measurement noise. In this case, instead of the form

$$y = (Cx + Du) + v,$$

where y is the measured signal and v is white Gaussian noise with power spectral density matrix V, we work with the integrated signal $z(t) = \int_0^t y(s)ds$, so that we can write again more rigorously

$$dz = (Cx + Du)dt + dB_2(t), \tag{3.9}$$

where B_2 is a Wiener process with incremental covariance $R_2 dt$. It is assumed that the processes B_2 and B_2 are independent.

Now assume that the process is sampled at discrete times $\{t_k\}_k$, and that we want as in the deterministic case to relate the values of x and z at the sampling times. Integrating (3.8) and denoting $h_k = t_{k+1} - t_k$, we get[6]

$$x(t_{k+1}) = e^{Ah_k} x(t_k) + \int_{t_k}^{t_{k+1}} e^{A(t_{k+1}-\tau)} Gud\tau + \int_{t_k}^{t_{k+1}} e^{A(t_{k+1}-\tau)} dB_1(\tau).$$

Assuming a fixed value $h_k = h$ for the intersample intervals and a ZOH, we get

$$x[k+1] = A_d x[k] + B_d u[k] + w[k],$$

where A_d and B_d are obtained as in the deterministic case, and the sequence $\{w[k]\}_k$ is a random sequence. This random sequence has the following properties, which come from the construction of stochastic integrals. First, the random variables $w[k]$ have zero mean

$$\mathbb{E}(w[k]) = \mathbb{E}\left(\int_{t_k}^{t_{k+1}} e^{A(t_{k+1}-\tau)} dB_1(\tau)\right) = 0,$$

[6] This can be admitted formally, by analogy with the deterministic case.

and a Gaussian distribution, which are general properties of these stochastic integrals. Their variance is equal to

$$\begin{aligned}\mathbb{E}(w[k]w[k]^T) &= \mathbb{E}\left(\int_{t_k}^{t_{k+1}} e^{A(t_{k+1}-\tau)}dB_1(\tau)\int_{t_k}^{t_{k+1}} dB_1^T(\tau')e^{A^T(t_{k+1}-\tau')}\right)\\ &= \int_{t_k}^{t_{k+1}}\int_{t_k}^{t_{k+1}} e^{A(t_{k+1}-\tau)}W\delta(\tau-\tau')e^{A^T(t_{k+1}-\tau')}d\tau d\tau'\\ &= \int_{t_k}^{t_{k+1}} e^{A(t_{k+1}-\tau)}We^{A^T(t_{k+1}-\tau)}d\tau. \end{aligned} \tag{3.10}$$

Here we have used the formal manipulations of Dirac deltas, but this formula is in fact a consequence of the *Ito isometry*. Finally, essentially by the independent increment property of the Brownian motion, we have that variables $w[k]$ and $w[k']$ for $k \neq k'$ are uncorrelated (or independent here since they are Gaussian)

$$\mathbb{E}(w[k]w[k']) = 0, \quad k \neq k'.$$

In other work, $\{w[k]\}_k$ is a discrete-time white Gaussian noise sequence with covariance matrix W_d given by (3.10), which we can rewrite

$$W_d = \int_0^h e^{At}We^{A^T t}dt,$$

by the change of variables $u = t_{k+1} - \tau$. Note in particular that

$$W_d \approx Wh$$

for h small.

Let us now consider the sampling of the stochastic measurement process (3.9). Note first that we have

$$\bar{y}[k+1] = z(t_{k+1}) - z(t_k) = \int_{t_k}^{t_{k+1}} y(\tau)d\tau, \tag{3.11}$$

which corresponds physically to the fact mentioned earlier that the random signal y containing high-frequency noise is not sampled directly but first integrated[7]. Thus we have

$$\begin{aligned}&\bar{y}[k+1] = z(t_{k+1}) - z(t_k)\\ &= \left(\int_{t_k}^{t_{k+1}} Ce^{A(t-t_k)}dt\right)x[k] + \left(\int_{t_k}^{t_{k+1}}\int_{t_k}^{t} Ce^{A(t-\tau)}d\tau dt\, B + Dh_k\right)u[k] + v[k],\\ &=: C_d x[k] + D_d u[k] + v[k],\end{aligned}$$

[7]Other forms of analog pre-filtering are possible and must be accounted for explicitly, by including a state-space model of the AAF filter. The simple integrate and reset filter (or average and reset) is the most commonly discussed in the literature on stochastic systems however.

where

$$v[k] = + \int_{t_k}^{t_{k+1}} \int_{t_k}^{t} Ce^{F(t-\tau)} dB_1(\tau) dt + B_2(t_{k+1}) - B_2(t_k).$$

Note first that the expressions for C_d, D_d can be rewritten

$$\begin{aligned}
C_d &= \int_0^h Ce^{At} dt \\
D_d &= \int_{t_k}^{t_{k+1}} \int_{\tau}^{t_{k+1}} Ce^{A(t-\tau)} dt d\tau B + Dh \\
&= \int_{t_k}^{t_{k+1}} \left(\int_0^{t_{k+1}-\tau} Ce^{As} ds \right) d\tau B + Dh \\
&=: \int_{t_k}^{t_{k+1}} \theta(t_{k+1} - \tau) d\tau B + Dh \\
&= \int_0^h \theta(u) du \, B + Dh.
\end{aligned}$$

Note the definition $\theta(t) := C \int_0^t e^{As} ds$. Similarly for $v[k]$ we have

$$v[k] = \int_{t_k}^{t_{k+1}} \theta(t_{k+1} - \tau) dB_1(\tau) + B_2(t_{k+1}) - B_2(t_k).$$

Again $\{v[k]\}$ is a sequence of Gaussian, zero-mean and independent random variables, i.e., discrete-time Gaussian white noise. We can also immediately compute [Åst06, p.83]

$$\begin{aligned}
\mathbb{E}(v[k]v[k]^T) &= \int_{t_k}^{t_{k+1}} \theta(t_{k+1} - \tau) W \theta(t_{k+1} - \tau) d\tau + Vh \\
&= \int_0^h \theta(s) W \theta^T(s) ds + Vh =: V_d \\
\mathbb{E}(w[k]v[k']^T) &= \delta[k - k'] \int_0^h e^{As} W \theta^T(s) ds =: \delta[k - k'] S_d.
\end{aligned}$$

Note in particular that the discrete samples $w[k]$ and $v[k]$ are not independent even if B_1 and B_2 are independent Wiener processes!

In summary, we obtain after integration of the output and sampling a stochastic difference equation of the form

$$x[k+1] = A_d x[k] + B_d u[k] + w[k] \tag{3.12}$$

$$\bar{y}[k+1] = C_d x[k] + D_d u[k] + v[k] \tag{3.13}$$

where the covariance matrix of the discrete-time noise can also be expressed as

$$\mathbb{E} \left\{ \begin{bmatrix} w[k] \\ v[k] \end{bmatrix} \begin{bmatrix} w[k] \\ v[k] \end{bmatrix}^T \right\} = \begin{bmatrix} W_d & S_d \\ S_d^T & V_d \end{bmatrix} = \int_0^h e^{\bar{A}t} \begin{bmatrix} W & 0 \\ 0 & V \end{bmatrix} e^{\bar{A}^T t} dt, \tag{3.14}$$

with

$$\bar{A} = \begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix} \Rightarrow e^{\bar{A}t} = \begin{bmatrix} e^{At} & 0 \\ C \int_0^t e^{A\tau} d\tau & I \end{bmatrix},$$

using Lemma 3.4.2.

Remark. Some authors assume that the measurements are of the averaging form

$$\bar{y}[k+1] = \frac{1}{h} \int_{t_k}^{t_{k+1}} y(t) dt,$$

instead of (3.11), see e.g. [GYAC10]. Then $\theta(t) \approx C$ for t close to zero, instead of $\theta(t) \approx Ct$ as $t \to 0$ here. This results in a covariance matrix where (3.14) should be multiplied on the left and right by $\texttt{blkdiag}(I, I/h)$. As a consequence of this choice however, the variance of the discrete-time measurement noise $v[k]$ diverges as $h \to 0$, and one should work with power spectral densities [GYAC10].

Remark. Note that in (3.13), there is a delay in the measurement, in contrast to the standard form of difference equations. Such a discrete-time delay is theoretically not problematic, since we can redefine the state as $\tilde{x}[k] = [x[k]^T, x[k-1]^T]^T$ and the measurement as $\tilde{y}[k] = \begin{bmatrix} 0 & C_d \end{bmatrix} \tilde{x}[k]$. Doubling the dimension of the state space has computational consequences however.

Nonlinear Systems

Poles and Zeros of Linear Sampled-Data Systems

Incremental Models

Choice of Sampling Frequency

Generalized Sample and Hold Functions

3.5 Discretization of Continuous-Time Controllers

3.6 Quantization Issues

3.7 Complements on Modern Sampling Theory and Reconstruction